Exam Analysis on Manifolds

WIANVAR-07.2015-2016.2A

April 4th, 2016, 9:00-12:00 hrs.

This exam consists of three assignments. You get 10 points for free.

Assignment 1. (30 pt.)
Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$, and let $i : M \to \mathbb{R}^3$ be the inclusion map.

1. (8 pt.) Construct an atlas on M such that:

   (i) M becomes a two-dimensional C^∞-manifold;
   (ii) the inclusion map i is a C^∞-map.

   Prove that this atlas satisfies both (i) and (ii).

2. (8 pt.) Let σ be the one-form on $\mathbb{R}^3$ given by $\sigma = x\,dx + y\,dy$.
   Prove that $i^*\sigma = 0$.

3. (7 pt.) Let $\Omega = dx \wedge dy \wedge dz$ be the volume form on $\mathbb{R}^3$, and let X be the vector field on $\mathbb{R}^3$ given by

   $$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

   Prove that $\iota_X\Omega = x dy \wedge dz - y dx \wedge dz$.
   (Recall that $\iota_X\Omega$ is the two-form given by $\iota_X\Omega(Y, Z) = \Omega(X, Y, Z)$.)

4. (7 pt.) Prove that $i^*(\iota_X\Omega)$ is a nowhere zero two-form on M.

Assignment 2. (30 pt.)
Let M be a compact connected $2n$-dimensional manifold without boundary, and let ω be a two-form on M.

In the following assignments, $\omega_k = \underbrace{\omega \wedge \cdots \wedge \omega}_{k \text{ factors}}$, for integers $k \geq 1$.

1. (8 pt.) Prove: if ω is closed, then for $k \geq 1$, the $2k$-form ω_k is also closed.

2. (8 pt.) Prove: if ω is exact, then for $k \geq 1$, the $2k$-form ω_k is also exact.

In the following two assignments, assume that ω_n is a nowhere zero $2n$-form on M.

3. (8 pt.) Prove that ω is not exact.

4. (6 pt.) Prove that M is not contractible to a point.

Assignment 3 on next page

Assignment 3. (30 pt.)
Let $M = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. We turn M into a Riemannian two-manifold by equipping it with the inner product $\langle \cdot, \cdot \rangle$, defined by

$$\langle v, w \rangle = \frac{v \cdot w}{h(x, y)^2},$$

for $v, w \in T_pM$, where $h : M \to \mathbb{R}$ is a positive C^2-function on M, and $v \cdot w$ is the standard inner product of v and w on $\mathbb{R}^2$.

1. (5 pt.) Prove that there is a differentiable function $f : M \to \mathbb{R}$ such that $\{F_1, F_2\}$, with $F_1 = f \frac{\partial}{\partial x}$ and $F_2 = f \frac{\partial}{\partial y}$, is an (orthonormal) moving frame on M.

2. (6 pt.) Determine the coframe $\{\vartheta_1, \vartheta_2\}$ of this moving frame (i.e., express these one-forms in terms of dx and dy).

3. (10 pt.) Determine the connection form ω_{12} of the moving frame $\{F_1, F_2\}$.

4. (9 pt.) Determine the Gaussian curvature $K(x, y)$ at $(x, y) \in M$, and prove that $K = -1$ for $h(x, y) = \frac{1}{2}(1 - x^2 - y^2)$.